# Advanced Logic
# Exam April 3, 2019: model answers

Rineke Verbrugge

19 June, 2020

# Exercise 1: Induction

Let $L_\leftrightarrow$ be an alternative language of propositional logic based on the operator $\leftrightarrow$ only (so without $\neg$, $\wedge$, $\vee$, $\rightarrow$).

(a) Give an inductive definition of the well-formed formulas of $L_\leftrightarrow$.

(b) Give an inductive definition of $v(A)$, the truth value of formula $A$ in the language $L_\leftrightarrow$ under valuation $v$.

(c) Let valuation $v_1$ be given such that $v_1(p) = 1$ for all propositional parameters $p$. Prove by induction that $v_1(A) = 1$ for all formulas $A$ in $L_\leftrightarrow$.

(d) Is $\{\leftrightarrow\}$ functionally complete, i.e. is it the case that every formula of propositional logic is equivalent to a formula in $L_\leftrightarrow$? Explain your answer.

# Answer 1: a, b

(a) Inductive definition of the well-formed formulas (wffs) of $L_{\leftrightarrow}$:

1. Each propositional parameter / atom $p_i$ is a wff of $L_{\leftrightarrow}$
2. If $A$ and $B$ are wffs of $L_{\leftrightarrow}$, then so is $(A \leftrightarrow B)$
3. Nothing is a wff of $L_{\leftrightarrow}$ if it cannot be constructed in finitely many iterations of steps 1 and 2.

(b) Inductive definition of the valuation function $v$:

1. $v(p_i)$ for all propositional parameters / atoms $p_i$ is defined
2. $v(A \leftrightarrow B) = 1- \mid v(A) - v(B) \mid$
   Also correct: $v(A \leftrightarrow B) =$
   $min(\{max(\{1 - v(A), v(B)\}, max(\{1 - v(B), v(A)\}\})\}$

# Answer 1, continued: c

(c) Proof by induction that for all wffs $A$ of $L_{\leftrightarrow}$, $v_1(C) = 1$:

BASIS: For each propositional parameter / atom $p_i$, $v_1(p_i) = 1$ by definition.

INDUCTIVE HYPOTHESIS: Suppose that for some arbitrary wffs $A$ and $B$ of $L_{\leftrightarrow}$, $v_1(A) = 1$ and $v_1(B) = 1$.

INDUCTIVE STEP Now $v_1(A \leftrightarrow B) = 1- \mid v_1(A) - v_1(b) \mid$ (see (b)). By the inductive hypothesis,
$1- \mid v_1(A) - v_1(b) \mid = 1- \mid 1 - 1 \mid = 1$.

CONCLUSION Therefore, for all wffs $C$ of $L_{\leftrightarrow}$, $v_1(C) = 1$

# Answer 1, continued: d

(d) No, $\{\leftrightarrow\}$ is *not* functionally complete. Take for example the formula $\neg p_1$. We have for the valuation $v_1$
$v_1(\neg p_1) = 1 - v_1(p_1) = 0$.

On the other hand, by (c) we have for all wffs $C$ of $L_{\leftrightarrow}$, $v_1(C) = 1$.

So there is no wff $C$ of $L_{\leftrightarrow}$ that is logically equivalent to $\neg p_1$ (i.e., that has the same truth value under *all* valuations, including $v_1$.

# Question 2

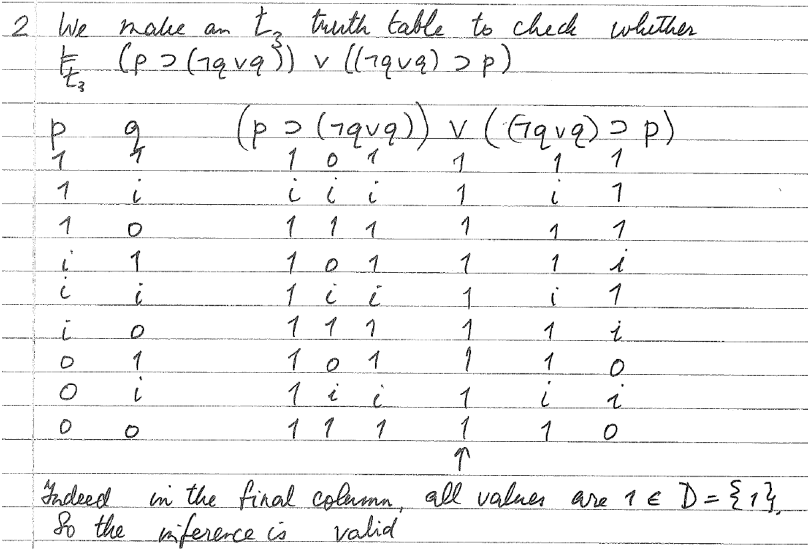
**Three-valued logics (10 pt)**

Using a truth table, determine whether the following inference holds in $\text{Ł}_3$:

$$\models_{\text{Ł}_3} (p \supset (\neg q \vee q)) \vee ((\neg q \vee q) \supset p)$$

Write out the full truth table and do not forget to draw a conclusion.

# Answer 2



2 We make an $Ł_3$ truth table to check whether $\models_{Ł_3} (p \supset (\neg q \vee q)) \vee ((\neg q \vee q) \supset p)$

| p | q | $(p \supset$ | $(\neg q$ | $\vee q))$ | $\vee$ | $((\neg q \vee q)$ | $\supset p)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | i | i | i | i | 1 | i | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| i | 1 | 1 | 0 | 1 | 1 | 1 | i |
| i | i | 1 | i | i | 1 | i | 1 |
| i | 0 | 1 | 1 | 1 | 1 | 1 | i |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | i | 1 | i | i | 1 | i | i |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
|  |  |  |  |  | ↑ |  |  |

Indeed in the final column, all values are $1 \in D = \{1\}$.
So the inference is valid

# Question 3

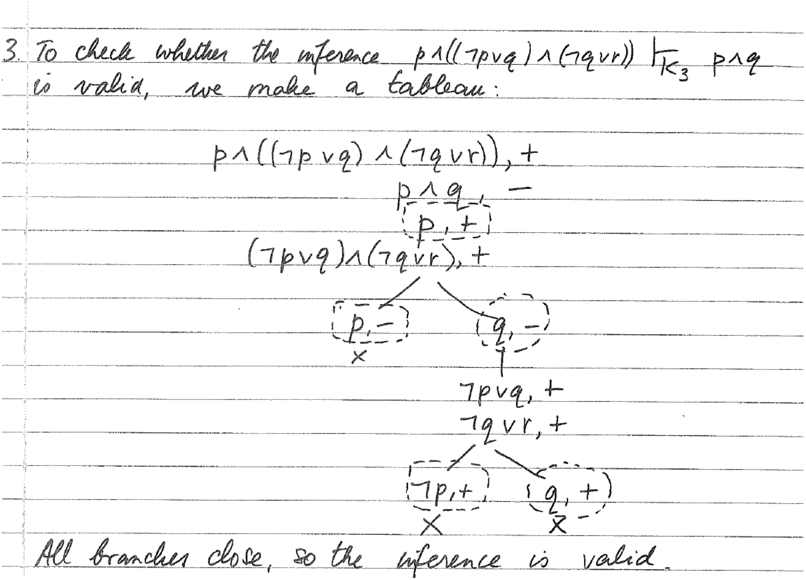
**Tableaux for FDE and related many-valued logics (10 pt)**

By constructing a suitable tableau, determine whether the following inference is valid in $\mathbf{K_3}$. If the inference is invalid, provide a counter-model.

$$p \wedge ((\neg p \vee q) \wedge (\neg q \vee r)) \vdash_{K_3} p \wedge q$$

NB: Do not forget to draw a conclusion from the tableau.

# Answer 3



3. To check whether the inference $p \wedge ((\neg p \vee q) \wedge (\neg q \vee r)) \vdash_{K_3} p \wedge q$ is valid, we make a tableau:

$p \wedge ((\neg p \vee q) \wedge (\neg q \vee r)), +$

$p \wedge q, -$

$\boxed{p, +}$

$(\neg p \vee q) \wedge (\neg q \vee r), +$

$\boxed{p, -}$ $\times$ $\quad$ $\boxed{q, -}$

$\neg p \vee q, +$

$\neg q \vee r, +$

$\boxed{\neg p, +}$ $\times$ $\quad$ $\boxed{q, +}$ $\times$

All branches close, so the inference is valid.

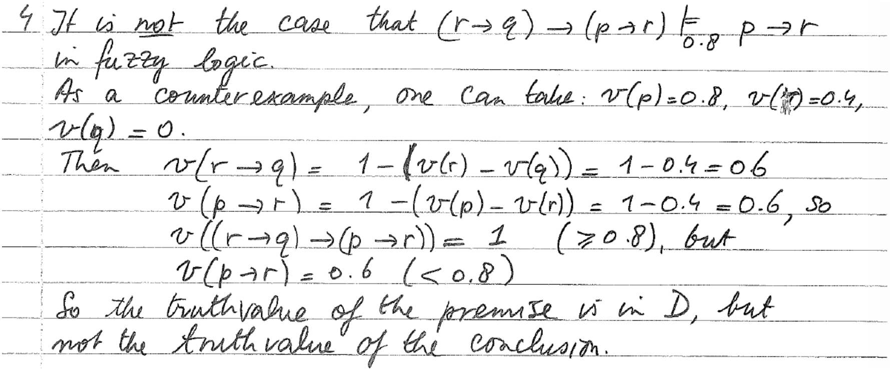
# Question 4

**Fuzzy logic (10 pt)**

Determine whether the following holds in the fuzzy logic with set of designated values $D_{0.8} = \{x : x \geq 0.8\}$. If so, explain why. If not, provide a counter-model and show why it is one.

$$(r \to q) \to (p \to r) \models_{0.8} p \to r$$

# Answer 4



4 It is not the case that $(r \to q) \to (p \to r) \models_{0.8} p \to r$ in fuzzy logic.

As a counterexample, one can take: $v(p)=0.8$, $v(r)=0.4$, $v(q)=0$.

Then $v(r \to q) = 1-(v(r)-v(q)) = 1-0.4 = 06$

$v(p \to r) = 1-(v(p)-v(r)) = 1-0.4 = 0.6$, so

$v((r \to q) \to (p \to r)) = 1 \quad (\geq 0.8)$, but

$v(p \to r) = 0.6 \quad (< 0.8)$

So the truthvalue of the premise is in D, but not the truthvalue of the conclusion.

# Question 5

**Basic modal tableau (10 pt)**

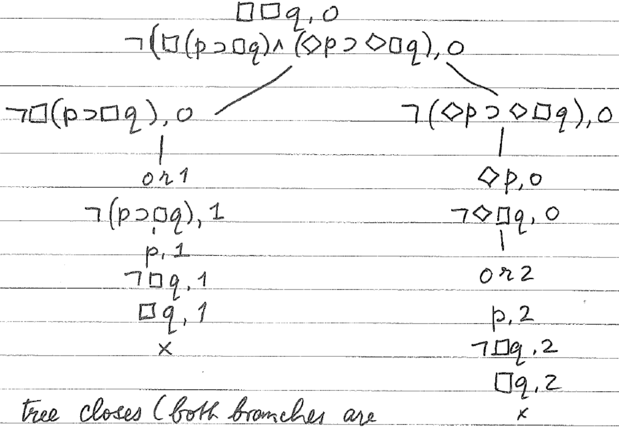
By constructing a suitable tableau, determine whether the following is valid in $K$. If the inference is invalid, provide a counter-model.

$$\Box\Box q \vdash_K \Box(p \supset \Box q) \wedge (\Diamond p \supset \Diamond\Box q)$$

NB: Do not forget to draw a conclusion from the tableau.

# Answer 5

5. To test whether the inference $\Box\Box q \vdash_k \Box(p \supset \Box q) \wedge (\Diamond p \supset \Diamond \Box q)$ is valid, we make a tableau:



The tree closes (both branches are closed), so the inference is valid.

# Question 6

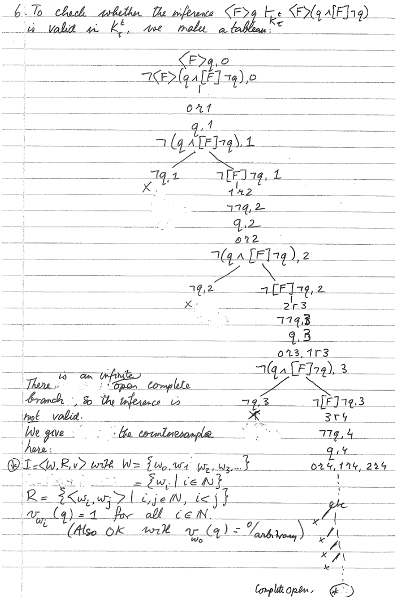
**Normal modal tableau (10 pt)**

By constructing a suitable tableau, determine whether the following tense-logical inference is valid in $K^t_\tau$ (transitive). If the inference is invalid, provide a counter-model.

$$\langle F \rangle q \ \vdash_{K^t_\tau} \langle F \rangle (q \wedge [F]\neg q)$$

NB: Do not forget to draw a conclusion from the tableau.

# Answer 6



6. To check whether the inference $\langle F\rangle q \vdash_{K^t_\tau} \langle F\rangle(q \wedge [F]\neg q)$ is valid in $K^t_\tau$, we make a tableau:

$\langle F\rangle q, 0$

$\neg\langle F\rangle(q \wedge [F]\neg q), 0$

$0r1$

$q, 1$

$\neg(q \wedge [F]\neg q), 1$

$\neg q, 1$ ×   |   $\neg[F]\neg q, 1$

$1r2$

$\neg\neg q, 2$

$q, 2$

$0r2$

$\neg(q \wedge [F]\neg q), 2$

$\neg q, 2$ ×   |   $\neg[F]\neg q, 2$

$2r3$

$\neg\neg q, 3$

$q, 3$

$0r3, 1r3$

$\neg(q \wedge [F]\neg q), 3$

$\neg q, 3$ ×   |   $\neg[F]\neg q, 3$

$3r4$

$\neg\neg q, 4$

$q, 4$

$0r4, 1r4, 2r4$

etc.

Complete open, ⊛

There is an infinite open complete branch, so the inference is not valid.

We give the counterexample here:

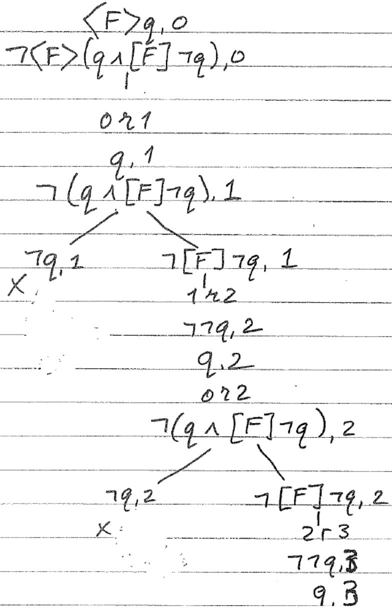
⊛ $I = \langle W, R, v\rangle$ with $W = \{w_0, w_1, w_2, w_3, \dots\}$
$= \{w_i \mid i \in N\}$

$R = \{\langle w_i, w_j\rangle \mid i, j \in N,\ i < j\}$

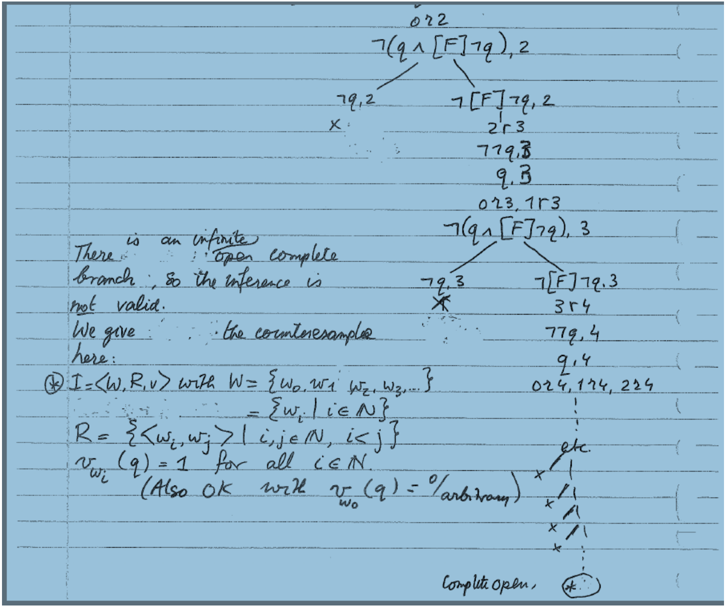
$v_{w_i}(q) = 1$ for all $i \in N$.

(Also OK with $v_{w_0}(q) = 0/\text{arbitrary}$)

# Answer 6 - zoomed in to top of tableau



# Answer 6 - zoomed in to bottom of tableau



There is an infinite open complete branch, so the inference is not valid.

We give the counterexample here:

(*) $I = \langle W, R, v \rangle$ with $W = \{w_0, w_1, w_2, w_3, \ldots\}$
$= \{w_i \mid i \in \mathbb{N}\}$

$R = \{\langle w_i, w_j \rangle \mid i, j \in \mathbb{N}, i < j\}$

$v_{w_i}(q) = 1$ for all $i \in \mathbb{N}$.

(Also OK with $v_{w_0}(q) = 0/\text{arbitrary}$)

# Question 7

**Soundness and completeness (10pt)**

As a reminder, the rule $\varphi$ for tense logic is:

$$\begin{tikzcd}
 & irj & \\
 & irk \arrow[dl] \arrow[d] \arrow[dr] & \\
jrk & j = k & krj
\end{tikzcd}$$

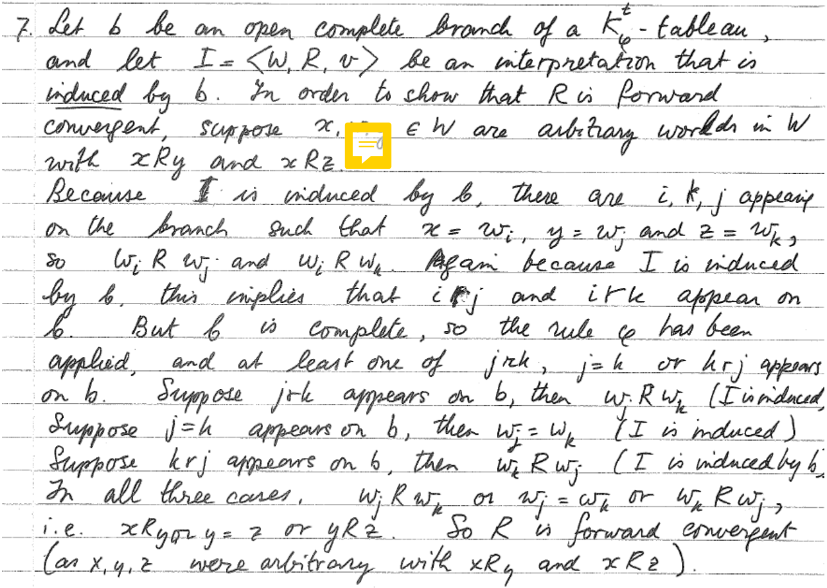
And the auxiliary rules for $=$ (that are included in the tense logic tableau system $K^t_\varphi$) are as follows, where $\alpha$ is a formula of the temporal language:

$$\begin{tikzcd}
\alpha(i) & \alpha(i) \\
i = j \arrow[d] & j = i \arrow[d] \\
\alpha(j) & \alpha(j)
\end{tikzcd}$$

# Question 7, continued

Let $b$ be a complete open branch of a $K^t_\varphi$-tableau, and let $I = \langle W, R, v \rangle$ be an interpretation that is *induced* by $b$. Show that the accessibility relation $R$ of $I$ is *forward convergent* , that is, for all $x, z, y \in W$, if $xRy$ and $xRz$, then ($zRy$ or $y = z$ or $yRz$).

# Answer 7



7. Let $b$ be an open complete branch of a $K^t_\varphi$-tableau, and let $I = \langle W, R, v \rangle$ be an interpretation that is induced by $b$. In order to show that $R$ is forward convergent, suppose $x,$ [illegible] $\in W$ are arbitrary worlds in $W$ with $xRy$ and $xRz$.

Because $I$ is induced by $b$, there are $i, k, j$ appearing on the branch such that $x = w_i$, $y = w_j$ and $z = w_k$, so $w_i R w_j$ and $w_i R w_k$. Again because $I$ is induced by $b$, this implies that $irj$ and $irk$ appear on $b$. But $b$ is complete, so the rule $\varphi$ has been applied, and at least one of $jrk$, $j=k$ or $krj$ appears on $b$. Suppose $jrk$ appears on $b$, then $w_j R w_k$ ($I$ is induced)

Suppose $j=k$ appears on $b$, then $w_j = w_k$ ($I$ is induced)

Suppose $krj$ appears on $b$, then $w_k R w_j$ ($I$ is induced by $b$)

In all three cases, $w_j R w_k$ or $w_j = w_k$ or $w_k R w_j$, i.e. $xRy$ or $y = z$ or $yRz$. So $R$ is forward convergent (as $x, y, z$ were arbitrary with $xRy$ and $xRz$).

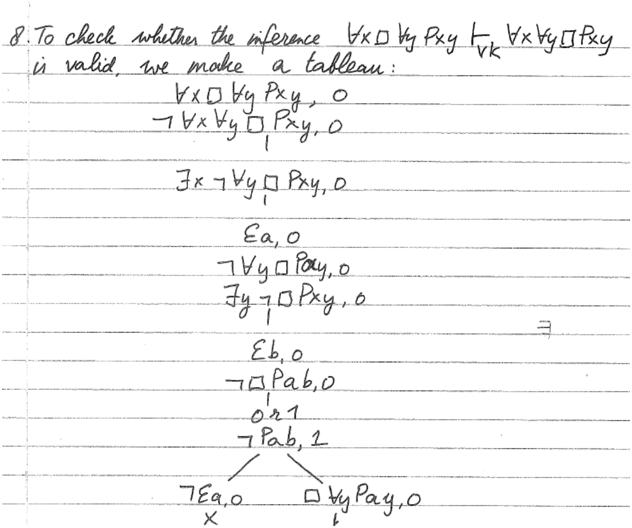
# Question 8

**First-order modal tableau, variable domain (10 pt)**
By constructing a suitable tableau, determine whether the following is valid in $VK$. If the inference is invalid, provide a counter-model.

$$\forall x \Box \forall y Pxy \vdash_{VK} \forall x \forall y \Box Pxy$$

NB: Do not forget to draw a conclusion from the tableau.

# Answer 8, top half



8. To check whether the inference $\forall x \Box \forall y\, Pxy \vdash_{VK} \forall x \forall y \Box Pxy$ is valid, we make a tableau:

$\forall x \Box \forall y\, Pxy,\ 0$

$\neg \forall x \forall y \Box Pxy,\ 0$

$\exists x \neg \forall y \Box Pxy,\ 0$

$\mathcal{E}a,\ 0$

$\neg \forall y \Box Pay,\ 0$

$\exists y \neg \Box Pxy,\ 0$

$\mathcal{E}b,\ 0$

$\neg \Box Pab,\ 0$

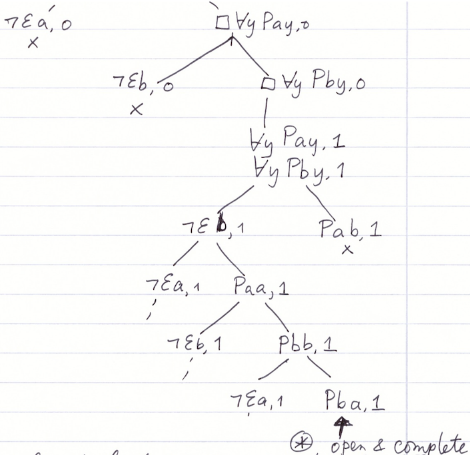
$0r1$

$\neg Pab,\ 1$

$\neg \mathcal{E}a,\ 0$ ×

$\Box \forall y\, Pay,\ 0$

# Answer 8, bottom half



The tableau has at least one open complete branch ⊛ so the inference is not valid. Counterexample from ⊛:

$I = \langle D, W, R, v \rangle$ with $D = \{\delta_a, \delta_b\}$, $W = \{w_0, w_1\}$,

$R = \{\langle w_0, w_1 \rangle\}$. $v_{w_0}(E) = \{\delta_a, \delta_b\} = D_{w_0}$

$v_{w_1}(E)$ can be chosen arbitrarily

$v_{w_1}(P) = \{\langle \delta_a, \delta_a \rangle, \langle \delta_b, \delta_b \rangle, \langle \delta_b, \delta_a \rangle\}$

# Question 9

**Default logic (10 pt)**

Consider the following set of default rules, where $p, q, r, s$ are propositional atoms:

$$D = \left\{ \delta_1 = \frac{p : q \wedge r}{s}, \qquad \delta_2 = \frac{p : q \wedge \neg r}{\neg r}, \qquad \delta_3 = \frac{s : \neg q}{\neg q} \right\},$$

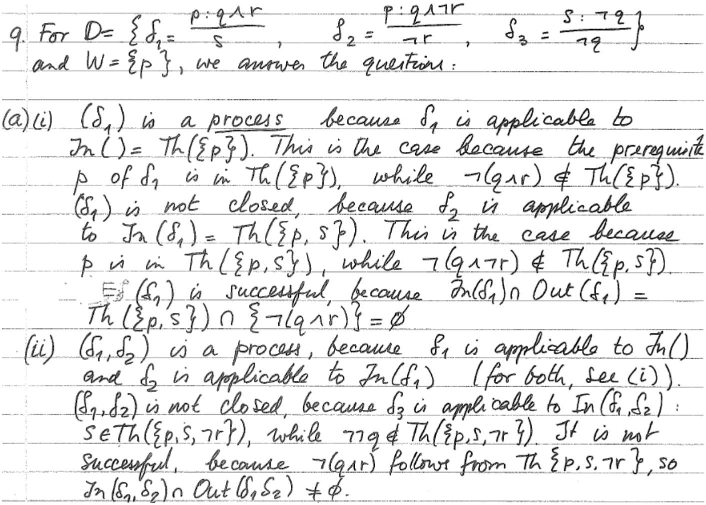
and initial set of facts:

$$W = \{p\}.$$

This exercise is about the default theory $T = (W, D)$.

1. Of each of the following sequences, state whether it is a *process*; and if so, whether or not the process is *closed*, and whether or not it is *successful*. Briefly explain your answers.
   1.1 $(\delta_1)$
   1.2 $(\delta_1, \delta_2)$
2. Draw the process tree of the default theory $(W, D)$.
3. What are the extensions of $(W, D)$?
4. Is $q \wedge s$ a credulous consequence of $(W, D)$? Explain.

# Answer 9a



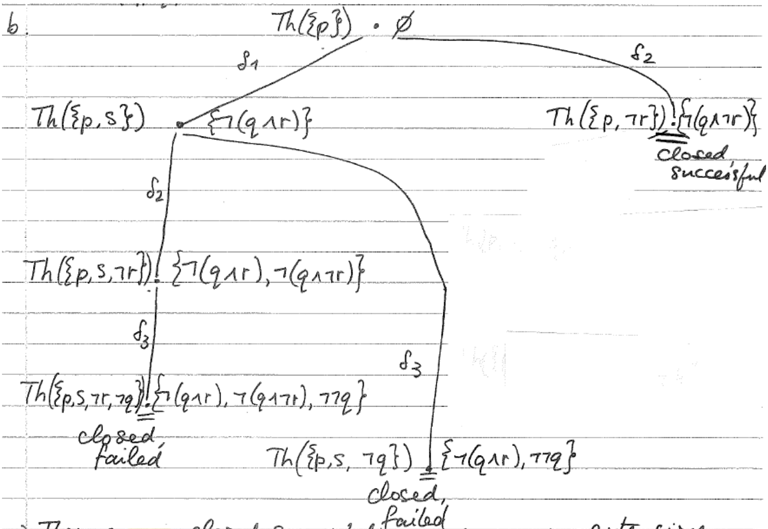
9. For $D = \{\delta_1 = \frac{p : q \wedge r}{s}, \quad \delta_2 = \frac{p : q \wedge \neg r}{\neg r}, \quad \delta_3 = \frac{s : \neg q}{\neg q}\}$ and $W = \{p\}$, we answer the questions:

(a)(i) $(\delta_1)$ is a <u>process</u> because $\delta_1$ is applicable to $In() = Th(\{p\})$. This is the case because the prerequisite $p$ of $\delta_1$ is in $Th(\{p\})$, while $\neg(q \wedge r) \notin Th(\{p\})$.
$(\delta_1)$ is not closed, because $\delta_2$ is applicable to $In(\delta_1) = Th(\{p, s\})$. This is the case because $p$ is in $Th(\{p, s\})$, while $\neg(q \wedge \neg r) \notin Th(\{p, s\})$.
$(\delta_1)$ is successful, because $In(\delta_1) \cap Out(\delta_1) = Th(\{p, s\}) \cap \{\neg(q \wedge r)\} = \emptyset$

(ii) $(\delta_1, \delta_2)$ is a process, because $\delta_1$ is applicable to $In()$ and $\delta_2$ is applicable to $In(\delta_1)$ (for both, see (i)).
$(\delta_1, \delta_2)$ is not closed, because $\delta_3$ is applicable to $In(\delta_1, \delta_2)$: $s \in Th(\{p, s, \neg r\})$, while $\neg\neg q \notin Th(\{p, s, \neg r\})$. It is not successful, because $\neg(q \wedge r)$ follows from $Th\{p, s, \neg r\}$, so $In(\delta_1, \delta_2) \cap Out(\delta_1, \delta_2) \neq \emptyset$.

# Answer 9 b,c,d

b.



c) There ar [illegible] closed successful branches, so no extensions

d) $q \wedge s$ is n[illegible] a credulous consequence of $(W, D)$, because there is no extension of which it is a member (see c: no extension at all)